9

TERMINOLOGY

expected value
exponential distribution
exponential random variable
Gaussian distribution
left and right rectangles
mean
median
midpoint rule
numerical integration
probability
probability density function
quantile
Simpson's rule
standard deviation
technology
trapezium rule
variance
solid of revolution

Prior learning

INTEGRATION AND APPLICATIONS OF INTEGRATION

APPLICATIONS OF INTEGRATION

APPLICATIONS OF INTEGRAL CALCULUS:

- the calculation of areas between curves determined by functions (ACMSM124)
- determine volumes of solids of revolution about either axis (ACMSM125)
- use numerical integration using technology (ACMSM126)
- use and apply the probability density function, $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$, of the exponential random variable with parameter $\lambda > 0$, and use the exponential random variables and associated probabilities and quantiles to model data and solve practical problems (ACMSM127) AC

9.01 AREAS BETWEEN CURVES

You can calculate the areas between the curves of known functions. For an enclosed area, it is often necessary for you to find the intersection points of the curves. The area can be calculated in two ways:

- as the difference between the areas under the two functions; or
- as the area under the **difference function**, because

$$\int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b [f(x) - g(x)]\, dx$$

Example 1

Find the area enclosed by the functions $f(x) = x^2$, $g(x) = (x - 4)^2$ and the x-axis.

Solution

Find the intersection(s). $x^2 = (x - 4)^2$

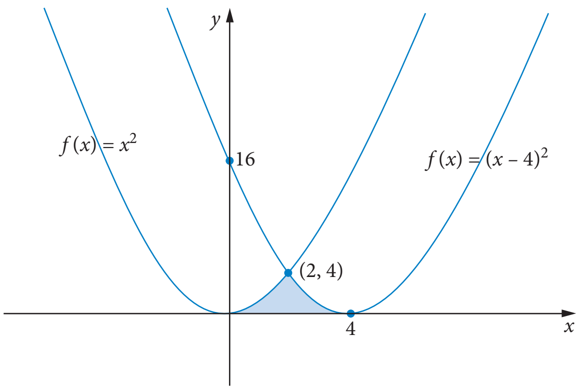
Simplify. $= x^2 - 8x + 16$

Solve for x. $x = 2$

Substitute to find y. $f(2) = 2^2 = 4$

Write the point of intersection. The graphs intersect at (2, 4).

Sketch the graphs.
Shade the required area.



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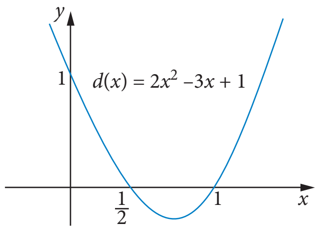
Write the integrals required.	Area $= \int_0^2 x^2 dx + \int_2^4 (x-4)^2 dx$
Simplify.	$= \int_0^2 x^2 dx + \int_2^4 (x^2 - 8x + 16) dx$
Calculate the integrals.	$= \left[\frac{x^3}{3}\right]_0^2 + \left[\frac{x^3}{3} - 4x^2 + 16x\right]_2^4$
	$= \left(\frac{8}{3} - 0\right) + \left(\frac{64}{3} - \frac{56}{3}\right)$
	$= 5\frac{1}{3}$
Write the answer.	The area enclosed by the functions $f(x) = x^2$, $g(x) = (x-4)^2$ and the x-axis is $5\frac{1}{3}$ square units.

For areas 'enclosed by' two functions, the difference method is generally better. You should do any algebraic simplification before integrating.

Example 2

Find the area enclosed by $y = 3x^2 - 4x - 1$ and $y = x^2 - x - 2$.

Solution

Find the difference function.	Let $d(x) = (3x^2 - 4x - 1) - (x^2 - x - 2)$
Simplify.	$= 2x^2 - 3x + 1$
Factorise.	$= (2x - 1)(x - 1)$
State the zeros.	The zeros are at $x = \frac{1}{2}$ and 1.
Sketch the graph of $d(x)$.	
Find the area needed.	Enclosed area $= -\int_{\frac{1}{2}}^{1} (2x^2 - 3x + 1) dx$
Calculate the integral.	$= -\left[\frac{2x^3}{3} - \frac{3x^2}{2} + x\right]_{\frac{1}{2}}^{1}$
Evaluate.	$= \frac{1}{24}$
Write the answer.	The area enclosed by $y = 3x^2 - 4x - 1$ and $y = x^2 - x - 2$ is $\frac{1}{24}$ square units.

In some cases, you will need to take account of the fact that part of the area is positive and part of it is negative in order to calculate the total area enclosed.

Example 3

Find the area enclosed by $y = x^3 + x^2 - 4x + 1$ and $y = 2x + 1$.

Solution

Find the difference function. Let $d(x) = (x^3 + x^2 - 4x + 1) - (2x + 1)$

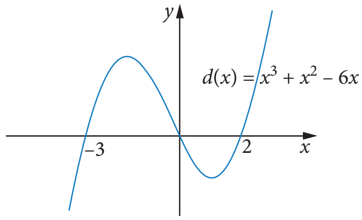
Simplify. $= x^3 + x^2 - 6x$

Find the zeros of $d(x)$. $= x(x^2 + x - 6)$

Factorise. $= x(x + 3)(x - 2)$

State the zeros. The zeros are at $x = -3$, 0 and 2.

Sketch the graph of $d(x)$.



Take account of the signs. Area enclosed $= \int_{-3}^{0}(x^3 + x^2 - 6x)dx - \int_{0}^{2}(x^3 + x^2 - 6x)dx$

Do the integrations. $= \left[\frac{x^4}{4} + \frac{x^3}{3} - 3x^2\right]_{-3}^{0} - \left[\frac{x^4}{4} + \frac{x^3}{3} - 3x^2\right]_{0}^{2}$

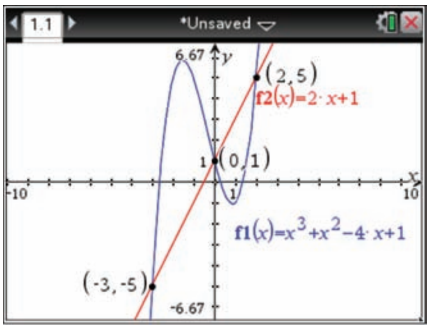
Do the calculations. $= \left[0 - \left(-\frac{63}{4}\right)\right] - \left(-\frac{16}{3} - 0\right)$

Simplify. $= \frac{63}{4} + \frac{16}{3} = 21\frac{1}{12}$

Write the answer. The area enclosed by $y = x^3 + x^2 - 4x + 1$ and $y = 2x + 1$ is $21\frac{1}{12}$ square units.

You would be wise to use your CAS calculator to check a problem such as Example **3**, even if you are required to do the problem by hand.

TI-Nspire CAS



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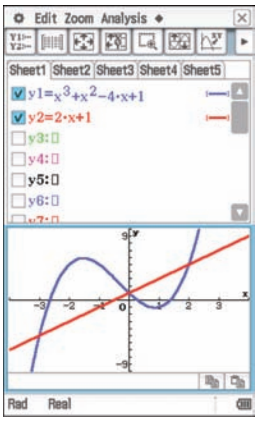

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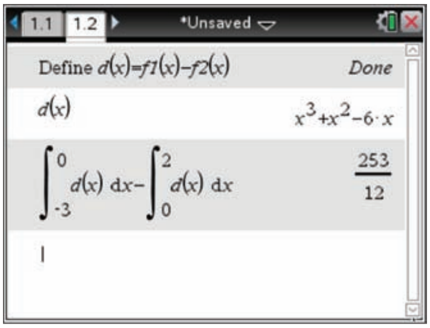

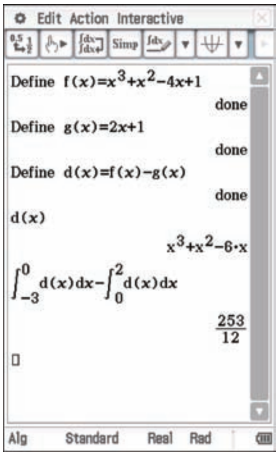

INVESTIGATION Lorenz curves

Not all Australians receive the same income. In fact, the poorest 10% of Australian families receive less than 2% of the total income, while the richest 10% receive 26% of the total income. *Distribution of income* is shown by a **Lorenz curve**, named after Konrad Lorenz who devised the method in 1905. The x-axis shows the proportion of the population whose incomes are the least. The y-coordinate of a point on the curve is the proportion of total income received by the proportion of the population shown on the x-axis. Since both proportions are between 0 and 1, the graph only goes from (0, 0) to (1, 1).

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If everyone received the same income, the graph would be the straight line $y = x$. The area between the curve and the line $y = x$ gives a measure of the income inequality of a society. In fact, the **coefficient of inequality** is defined to compare the inequality of the distribution from one society to another.

The coefficient of inequality is given by

$$L = \frac{\text{area between Lorenz curve and the line } y = x}{\text{area under the line } y = x}$$

In 2011–2012, the net worth of Australian households was as follows:

Poorest 20%:	\$31 203
Next 20%:	\$191 207
Middle 20%:	\$437 856
Next 20%:	\$766 465
Wealthiest 20%:	\$2 215 032

Source: ABS 6523.0 Table 7

Assuming the cumulative incomes at x-values of 0.2, 0.4, 0.6, 0.8 and 1, find the total wealth by adding the amounts and then find the ratio to the total of the cumulative amounts.

Draw the Lorenz curve and use your calculator to find a line of best fit. Remember to include the points (0, 0) and (1, 1).

Calculate the coefficient of income inequality for Australia in 2011–2012.

Find data for earlier years and look at trends in Australia. Research income inequality in other countries. What do you find?

EXERCISE 9.01 Areas between curves

Concepts and techniques

WS Areas between curves

WS Areas between curves (2)

1 Example 1 Find the area enclosed by the functions $f(x) = (2 - x)^3$, $g(x) = x^3$ and the x-axis.

2 Find the area enclosed by the functions $f(x) = x^2 - 4$, $g(x) = 2x + 4$, the line $x = 1$ on the left and the x-axis.

3 Example 2 Find the area enclosed by $y = 3x^2 - 2x - 3$ and $y = 2x^2 + 3x + 3$.

4 Find the area enclosed by $y = 3x^2 - 3x - 1$ and $y = 2x^2 - x + 7$.

5 Find the area enclosed by $y = 2x^2 + 3x + 2$ and $y = x^2 + 5x + 10$.

6 Find the area enclosed by $y = 2x^2 + 12x - 16$ and $y = x + 4 - x^2$.

7 Example 2 Find the area enclosed by $y = 2x^3 + 5x^2 - 9x + 5$ and $y = 3x + 5$.

8 Find the area enclosed by $y = x^3 - 4x^2 - 2x + 11$ and $y = 2x - 5$.

Reasoning and communication

9 In Malaysia, the Lorenz curve for income is approximately

$$y = 0.06x + 0.18x^2 + 0.17x^3 + 0.35x^4 + 0.24x^5$$

a Sketch the Lorenz curve.

b What proportion of income is earned by the poorest 10% of the population?

c What proportion of income is earned by the wealthiest 10% of the population?

d What is the coefficient of income inequality in Malaysia?

10 Find the area enclosed by $y = x^4 + 5x^3 + 7x^2 - 9$ and $y = 2x^2 + 5x - 3$.

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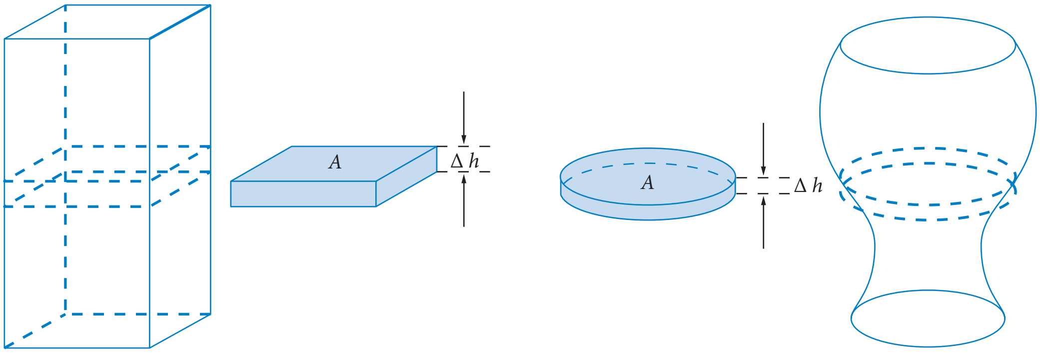
9.02 VOLUMES OF SOLIDS OF REVOLUTION

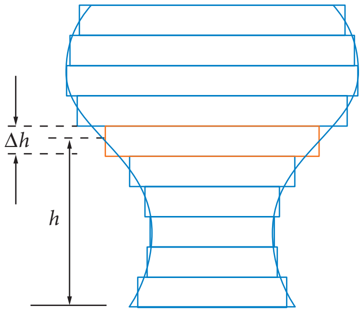
You can approximate the volumes of three-dimensional shapes by slicing them into layers, finding the volume of each layer, and adding your answers together. The volume of a single slice through a three-dimensional shape is $A\Delta h$, where A is the area of the slice and Δh is the thickness of the slice, as shown below.



If successive slices of the same thickness were taken all the way up an object, the total volume would be approximately $V \approx \sum A\Delta h$ where the sum is taken over the height.

The area of each slice actually depends on its position. It would be more correct to write $A = f(h)$, since the area of the slice is a function of h. In order to obtain a more accurate value of V, make the slices thinner. As $\Delta h \to 0$, the approximation to the volume will become exact. In this case, the area of the slice becomes the cross-sectional area at the 'height' h.

Then	$V \approx \sum A\Delta h$
But $A = f(x)$, so	$V \approx \sum f(h)\Delta h$
In the limit	$V = \lim\limits_{\Delta h \to 0} \sum f(h)\Delta h$



But this is the definite integral, so

$$V = \int f(h)dh = \int A(h)dh = \int A\,dh.$$

The function integrated is the cross-sectional area at x, which can be written as $A(h)$.

IMPORTANT

Volume of a shape

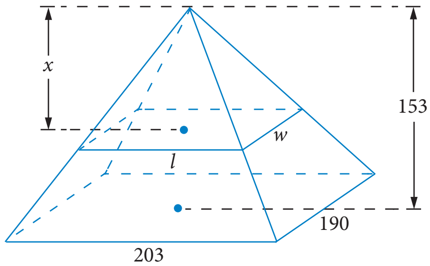
$V = \int A(h)dh = \int A\,dh$

Example 4

Use integration to find the volume of a pyramid of height 153 m with a rectangular base 203 m by 190 m.

Solution

Draw a diagram.
Measure the 'height' from the top of the pyramid to the base, as shown.



Write ratios to find l and w at height x.

$l = \frac{x}{153} \times 203$ and $w = \frac{x}{153} \times 190$

Find an expression for the cross-sectional area.

$A = lw$

Write the area formula at x.
Substitute and simplify.

$$= \frac{x}{153} \times 203 \times \frac{x}{153} \times 190$$
$$= \frac{203 \times 190}{153^2} x^2$$

Write the volume formula.

$$V = \int A(x)\,dx$$

Integrate from $x = 0$ to $x = 153$ to find the volume.

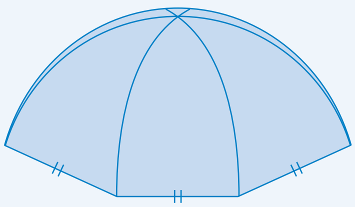
$$V = \int_0^{153} \frac{203 \times 190}{153^2} x^2\,dx$$
$$= \frac{203 \times 190}{153^2} \int_0^{153} x^2\,dx$$
$$= \frac{203 \times 190}{153^2} \left[\frac{x^3}{3}\right]_0^{153}$$
$$= \frac{203 \times 190 \times 153}{3}$$
$$= 1\,967\,070 \text{ m}^3$$

Write the answer.

The volume is 1 967 070 m^3.

INVESTIGATION Pop-up tent

A lightweight 'pop-up' tent consists of six plastic struts that are inserted into pockets sewn into the joins of the fabric panels. The resulting shape has hexagonal horizontal cross-sections, while vertical cross-sections through the centre are semicircular. The overall height is 1.3 m. In order to determine whether it is safe to use a gas lamp inside, a camper needs to know the volume of air in the tent. Investigate the volume inside the tent.



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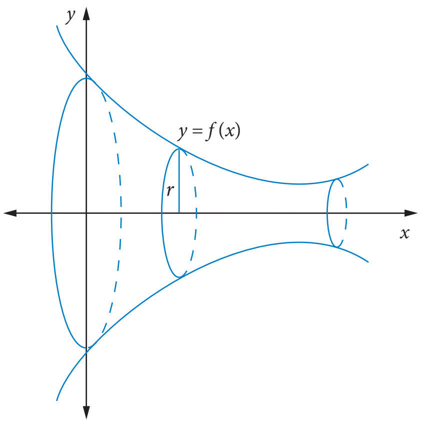
Many objects have a circular cross-section because they are manufactured by *turning* on a lathe. Objects of this shape are called **solids of revolution**, and their volumes can be found by using integration. Consider the axis of revolution as being the x-axis and write the radius as $y = f(x)$.



Then $y = f(x)$ is the radius of the cross-section, and the cross-section is a circle.

Thus $A = \pi r^2 = \pi y^2$.

The volume of a solid of revolution is given by

$V = \int A\,dx = \int \pi y^2\,dx = \pi \int y^2\,dx$.

IMPORTANT

Volumes of solids of revolution

The volume of a solid of revolution generated by a curve in the xy plane is given by

$V = \pi \int y^2\,dx$ when rotated about the x-axis

or $V = \pi \int x^2\,dy$ when rotated about the y-axis.

Example 5

Find the volume of the bell of a French horn that could be formed by rotating the curve $y = 0.05x^3$ about the x-axis between $x = 0$ and $x = 5$. Measurements are in centimetres.

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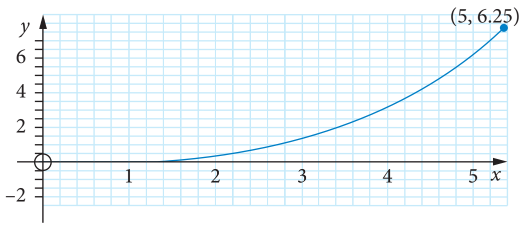
Solution

Draw a diagram.



Write the formula for volume rotated about the x-axis between $x = 0$ and $x = 5$.

$V = \pi \int_0^5 y^2\,dx$

Find an expression for y^2.

$y^2 = (0.05x^3)^2$

$= 0.0025x^6$

Substitute into the integration formula.

$V = \pi \int_0^5 0.0025x^6\,dx$

Integrate.	$= 0.0025\pi\left[\frac{1}{7}x^7\right]_0^5$
Evaluate.	$= 0.0025\pi\left(\frac{1}{7}\cdot 5^7\right)$
	$= \dfrac{3125\pi}{112}$
Write the answer.	Volume ≈ 87.656 cm^3.

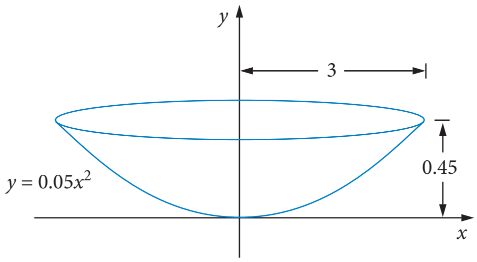
You can choose the y-axis as the axis of rotation; in this case you would swap the integration variables for y-values and find an expression for x^2.

Example 6

Find the volume of a parabolic telescope mirror of radius 3 m formed by rotating the curve $y = 0.05x^2$ about the y-axis.

Solution

Draw a diagram.



The limits of integration will be the values of y that give the correct range of x. Substitute $x = 0$ and $x = 3$.	$x = 0$ gives $y = 0$. $x = 3$ gives $y = 0.05x^2$ $= 0.05 \times 3^2$ $= 0.45$
Write the formula for volume rotated about the y-axis between $y = 0$ and $y = 0.45$.	$V = \pi\int_0^{0.45} x^2 dy$
Find an expression for x^2.	$y = 0.05x^2$ $\therefore x^2 = 20y$
Substitute into the integration formula.	$V = \pi\int_0^{0.45} 20y\, dy$
Integrate.	$= \pi\left[10y^2\right]_0^{0.45}$
Evaluate.	$= \dfrac{81\pi}{40} \approx 6.36\,\text{m}^3$
Write the answer.	The volume is about 6.36 m^3.

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EXERCISE 9.02 Volumes of solids of revolution

Concepts and techniques

WS Solids of revolution

1 Example 5 Use a suitable definite integral to find the volumes, in cubic units, that are formed by rotating the following curves about the x-axis between the limits shown.

a $y = 0.5x^2$, from $x = 0$ to $x = 3$.

b $y = 5x^3$, from $x = 0$ to $x = 1$.

c $y = \log_e(x)$, from $x = 1$ to $x = 2$.

d $y = \frac{1}{2}e^{\frac{1}{2}x}$, from $x = 1$ to $x = 2$.

e $y = \frac{1}{2}\cos\left(\frac{1}{2}x\right)$, from $x = \pi$ to $x = 2\pi$.

2 Example 6 Use a suitable definite integral to find the volumes, in cubic units, that are formed by rotating the following curves about the y-axis between the limits shown.

a $y = 0.5x^2$, from $y = 0$ to $y = 3$

b $y = 5x^3$, from $y = 0$ to $y = 1$

c $y = \log_e(x)$, from $y = 0$ to $y = 1$

d $y = \frac{1}{2}e^{\frac{1}{2}x}$, from $x = 1$ to $x = 2$

e $y = \frac{1}{2}\cos\left(\frac{1}{2}x\right)$, from $x = 0$ to $x = \pi$

f $y = \sqrt{x} + 1$, from $x = 1$ to $x = 2$

3 a A sphere of radius 5 cm can be considered to be the solid of revolution of the curve $y = \sqrt{25 - x^2}$ around the x-axis. Using integration, the volume of the hemisphere is

A $\frac{25\pi}{3}$ B 25π C 100π D $\frac{500\pi}{3}$ E 500π

b What formula could be used to get the same result?

4 The inside of a pottery urn that a small child could hide in can be modelled as a solid of revolution of the line $y = 0.000\,05x^3 - 0.0045x^2 + 20$ cm around the x-axis from $x = 0$ to $x = 80$ cm, with the open end at $x = 0$ and a flat base. The maximum volume, in cubic cm, of oil that could be held by the urn when it is full, is closest to

A 42.22 B 71.817 C 4222 D 22 860 E 71 817

5 A pointed hat is modelled by rotating the line $y = \sqrt{0.2x}$ from $x = 0$ to $x = 20$ about the y-axis. If the measurements are in cm, the volume of the hat is

A 40 B $\frac{80\pi}{3}$ C 160π D $\frac{1600}{3}$ E $\frac{1600\pi}{3}$

6 The interior surface of an hourglass can be considered as having the curve $y = \sqrt{1.05 - \frac{1}{1 + x^2}}$ rotated about the x-axis between $x = -5$ and $x = 5$ (measurements are in cm). The volume of the hourglass is closest to

A $\tan^{-1}(x)$ B 24.36 C $\tan^{-1}\left(\frac{x}{1.05}\right)$ D 26.88 E 24.36π

Reasoning and communication

7 Example 4 Find the formula for the volume of a square-based pyramid of side s and height h.

8 Use integration to find the volume of a pyramid of height 15 m with a rectangular base measuring 30 m by 40 m.

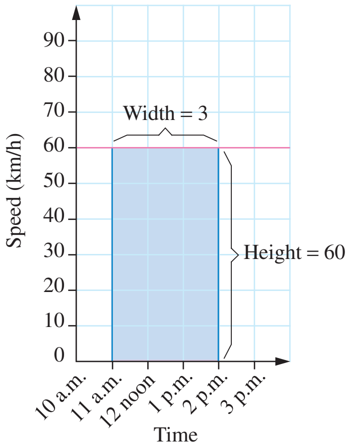
9 Use integration to find the volume of a cone of height 10 cm with a base diameter of 15 cm.

10 The interior surface of a glass *objet d'art* is formed from a molten glass sheet by allowing it to sag through a circular opening. The resulting cross-sectional shape may be modelled by the *catenary* curve $f(x) = 15.43 - 5(e^{0.05x} + e^{-0.05x})$ between $x = -20$ and $x = 20$, where all measurements are in cm. Use rotation about the x-axis between $x = -20$ and $x = 20$ to find the interior volume of the *objet d'art*.

Alamy/Radius Images

9.03 NUMERICAL INTEGRATION

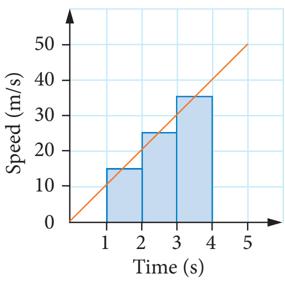
The graph on the right shows the speed of a car travelling along a main road with little traffic. Between 11 a.m. and 2 p.m. it travels for 3 hours at 60 km/h, so the distance is $60 \times 3 = 180$ km. On the graph, this is shown by the area of the rectangle. Its width is 3 (hours) and its height is 60 (km/h), so its area is $60 \times 3 = 180$ (km).

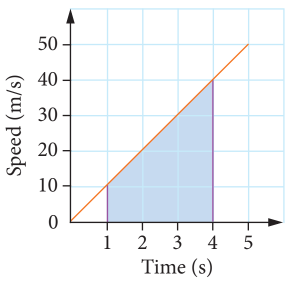


Suppose that a drag-racing car smoothly accelerated from 0 to 50 m/s (180 km/h) in 5 seconds. In the 2nd second it would accelerate from 10 to 20 m/s, so its average speed would be 15 m/s and it would travel 15 m in that second. In the 3rd second it would travel 25 m, and so on.

The first graph below shows rectangles. Each rectangle's height is the average speed for that second. The area of each rectangle is the distance covered in that second. The total of the areas of the rectangles is the total distance covered from 1 to 4 seconds.

The second graph shows a trapezium under the line. This trapezium has the same area as the total area of all the rectangles in the first graph, so its area is also the distance travelled from 1 to 4 seconds.





The area under a speed–time graph is the distance covered by an object during the time shown. You can find areas under curves (definite integrals) using a range of approximation techniques. Technology can be used to help you find these answers. The first graph illustrates the simplest such method, the **midpoint rule**.

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IMPORTANT

The **midpoint rule** is that $\int_a^b f(x)dx \approx w\sum_{i=1}^{n} f(a_i)$, where the interval $[a, b]$ is divided into n equal width strips of width w and the values $a_1, a_2, \ldots, a_n$ are the midpoints of the strips.

The rule can also be written as $\int_a^b f(x)dx \approx w\sum_{i=1}^{n} f\left(\frac{a_{i-1}+a_i}{2}\right)$, where $a_0, a_1, a_2, \ldots, a_n$ are the endpoints of the strips, so $a_0 = a$ and $a_n = b$.

Example 7

Find an approximation for the definite integral of $f(x) = \log_e(5x)$ from $x = 2$ to $x = 6$ by calculating the areas of strips 0.5 units wide, using the midpoint rule.

Solution

Divide into strips.

The strips are [2, 2.5], [2.5, 3], …, [5.5, 6].

Find the midpoints of the vertical strips.

The centres are 2.25, 2.75, …, 5.75.

Write a rule for the midpoints.

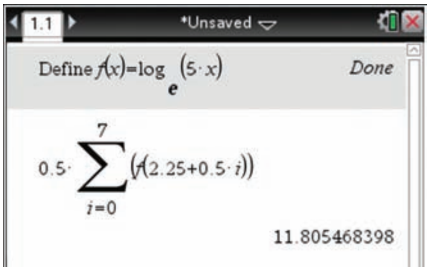
Midpoints = 2.25 + 0.5i for i = 0 to 7.

Write the required approximation.

$$\int_2^6 \log_e(5x)dx \approx 0.5\times\sum_{i=0}^{7} f(2.25+0.5i)$$

TI-Nspire CAS

Define the function. Make sure that you use the constant e, not the letter e.
Use the template menu [template key] to get the sum to find the approximation.



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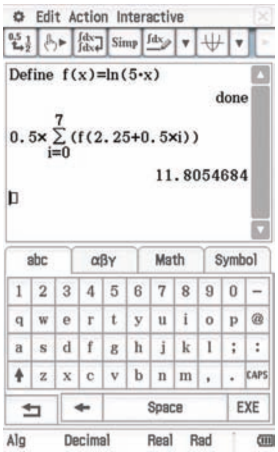
Define the function. Make sure that you use [ln] from [Math1] for $\log_e$.

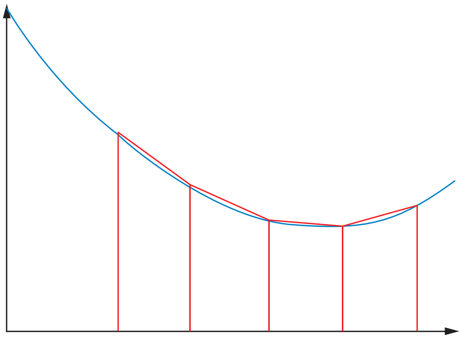
Enter $0.5\times\sum_{i=0}^{7} f(2.25+0.5i)$ using [Σ template] from [Math2]. Make sure that you use i from the [abc] menu, not i for imaginary numbers.



Round and write the answer.

$$\int_2^6 \log_e(5x)dx \approx 11.805$$

Instead of using rectangles, a better approximation can be obtained using trapeziums. This means that the tops are sloping, so they approximate the function a little better.



The area of a trapezium is given by $\frac{1}{2}h(a+b)$. For n strips of width w between a and b with endpoints a_0, $a_1, a_2, \ldots, a_n$ the areas of the strips will be

$\frac{1}{2}w[f(a_0)+f(a_1)], \frac{1}{2}w[f(a_1)+f(a_2)],$

$\frac{1}{2}w[f(a_2)+f(a_3)], \ldots, \frac{1}{2}w[f(a_{n-2})+f(a_{n-1})],$

$\frac{1}{2}w[f(a_{n-1})+f(a_n)]$ so the total area of the strips will be given by

$$\frac{1}{2}w[f(a_0)+f(a_1)]+\frac{1}{2}w[f(a_1)+f(a_2)]+\ldots+\frac{1}{2}w[f(a_{n-2})+f(a_{n-1})]+\frac{1}{2}w[f(a_{n-1})+f(a_n)]$$

$$=\frac{1}{2}w[f(a_0)+f(a_1)+f(a_1)+f(a_2)+\ldots+f(a_{n-2})+f(a_{n-1})+f(a_{n-1})+f(a_n)]$$

$$=\frac{1}{2}w[f(a_0)+2f(a_1)+2f(a_2)+\ldots+2f(a_{n-1})+f(a_n)]$$

This is called the **trapezoidal rule**.

IMPORTANT

The **trapezoidal rule** is that $\int_a^b f(x)dx \approx \frac{w}{2}\left[f(a_0)+2\sum_{i=1}^{n-1} f(a_i)+f(a_n)\right]$, where the interval $[a, b]$ is divided into n equal width strips of width w and end values $a_0, a_1, \ldots, a_n$. Of course $a_0 = a$ and $a_n = b$.

The rule can also be written as $\int_a^b f(x)dx \approx \frac{w}{2}(E+2M)$, where E is the sum of the values of $f(x)$ at the ends of the interval and M is the sum of all the values of $f(x)$ at the intervening points $a_1, a_2, \ldots, a_{n-1}$.

Example 8

Use the trapezium rule to find an approximation for $\int_{-2}^{2} e^x dx$ using 8 strips.

Solution

Find the width of each strip. Width of strips $=\frac{2-(-2)}{8}=\frac{1}{2}$

Find the points. Endpoints are $-2, -1.5, -1, \ldots, 2$.

Write the points at the ends of the interval. $a_0 = -2$, $a_8 = 2$

Write a formula for the remaining points. $a_i = -2 + 0.5i$ for $i = 1$ to 7

Write the required approximation.

$$\int_{-2}^{2} e^x dx$$

$$\approx \frac{0.5}{2}\left[f(-2)+2\sum_{i=1}^{7} f(-2+0.5i)+f(2)\right]$$

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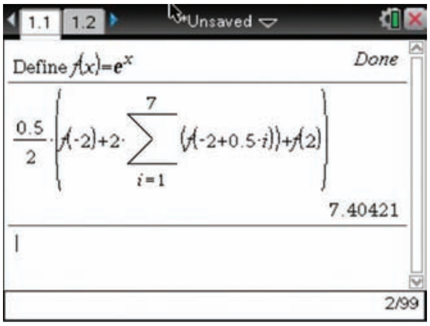
TI-Nspire CAS

Define the function. Make sure you use the constant e, not the letter e.
Use the template menu [template] to get the sum to find the approximation.

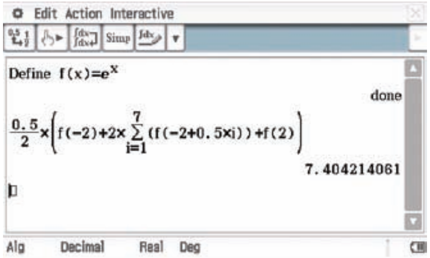


ClassPad

Define the function. Make sure that you use $e^\blacksquare$ from Math1 for e^x.

Enter $\frac{0.5}{2}\left[f(-2)+2\sum_{i=1}^{7} f(-2+0.5i)+f(2)\right]$

using Σ from Math2. Make sure that you use i from the abc menu, not i for imaginary numbers.



Round and write the answer.

$\int_{-2}^{2} e^x dx \approx 7.4042$

A further improvement to the approximation of integrals using equal width strips can be made by using *double strips* with parabolas at the top instead of straight lines. In this case, for an even number of strips with endpoints $a_0, a_1, a_2, \ldots, a_n$, you can show that the areas of the strips are given by $\frac{w}{3}(a_0 + 4a_1 + a_2), \frac{w}{3}(a_2 + 4a_3 + a_4), \frac{w}{3}(a_4 + 4a_5 + a_6), \ldots, \frac{w}{3}(a_{n-2} + 4a_{n-1} + a_n)$. Adding the areas of these strips gives **Simpson's rule**.

IMPORTANT

Simpson's rule is that $\int_a^b f(x)dx \approx \frac{w}{3}\left[f(a_0)+2\sum_{i=1}^{\frac{n}{2}-1} f(a_{2i})+4\sum_{i=1}^{\frac{n}{2}} f(a_{2i-1})+f(a_n)\right]$, where the interval $[a, b]$ is divided into an even number n of equal width strips of width w and end values $a_0, a_1, \ldots, a_n$. Of course $a_0 = a$ and $a_n = b$.

The rule can also be written as $\int_a^b f(x)dx \approx \frac{w}{3}(X + 2E + 4O)$, where X is the sum of the values of $f(x)$ at the ends of the interval, E is the sum of the values at the even-numbered intervening points $a_2, a_4, \ldots, a_{n-2}$ and O is the sum of the values at the odd-numbered intervening points $a_1, a_3, \ldots, a_{n-1}$.

Example 9

Use Simpson's rule to find $\int_1^5 \frac{5}{x}dx$ using strips of width $\frac{1}{4}$.

Solution

Find the points. Points are 1, 1.25, 1.5, …4.5, 4.75, 5

Write the points at the ends of the interval. $a_0 = 1$, $a_{16} = 5$

Write a formula for the remaining odd points. $a_i = 1.25 + 0.5i$ for $i = 0$ to 7

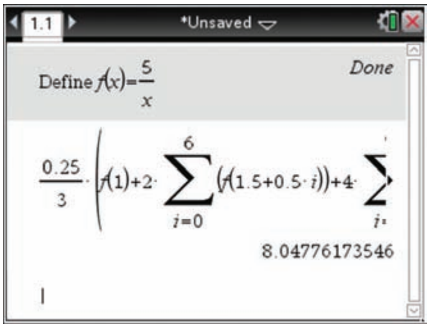
Write a formula for the remaining even points. $a_i = 1.5 + 0.5i$ for $i = 0$ to 6

Write the required approximation.

$$\int_1^5 \frac{5}{x}dx \approx \frac{0.25}{3}\left[f(1)+2\sum_{i=0}^{6} f(1.5+0.5i)+4\sum_{i=0}^{7} f(1.25+0.5i)+f(5)\right]$$

TI-Nspire CAS

Define the function.
Use the template menu to get the sums to find the approximation.

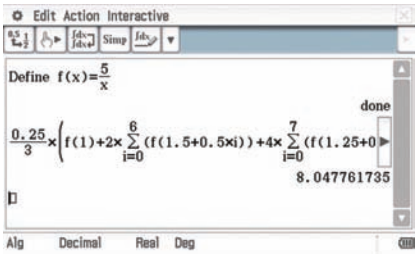


ClassPad

Define the function.
Enter

$$\frac{0.25}{3}\left[f(1)+2\sum_{i=0}^{6} f(1.5+0.5i)+4\sum_{i=0}^{7} f(1.25+0.5i)+f(5)\right]$$

using Σ from Math2. Make sure that you use i from the abc menu, not *i* for imaginary numbers.



Round and write the answer. $\int_1^5 \frac{5}{x}dx \approx 8.0478$

For a cubic, quadratic or straight line, Simpson's rule actually gives the exact answer. Your CAS calculator uses methods that are more advanced than Simpson's rule to find approximations for integrals that cannot be integrated algebraically.

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EXERCISE 9.03 Numerical integration

Concepts and techniques

WS Numerical integration

WS Trapezoidal rule

1 Example 7 CAS An approximation for the definite integral of $f(x)=\frac{1}{2}x^2$, from $x=0$ to $x=4$ with strips 1 unit wide, using the midpoint rule is

A 7 B 10 C $10\frac{1}{2}$ D $10\frac{2}{3}$ E $12\frac{1}{8}$

2 CAS An approximation for the area under the curve $y=\frac{1}{2}x^2$ from $x=0$ to $x=4$ with strips $\frac{1}{2}$ a unit wide, using the midpoint rule is

A $5\frac{1}{3}$ B 6.0625 C $10\frac{1}{2}$ D 10.625 E $10\frac{2}{3}$

3 Example 8 CAS An approximation for the area under the curve $y=\frac{1}{2}x^2$ from $x=0$ to $x=1$ with strips a $\frac{1}{4}$ unit wide, using the trapezoidal rule is

A $\frac{11}{64}$ B $\frac{21}{128}$ C $\frac{1}{6}$ D 10.625 E $10\frac{2}{3}$

4 CAS An approximation for the area under the curve $y=\frac{1}{2}x^2$ from $x=0$ to $x=4$ with strips a $\frac{1}{4}$ unit wide, using the trapezoidal rule is

A $\frac{5}{64}$ B $\frac{21}{128}$ C $\frac{1}{6}$ D $\frac{13}{64}$ E $10\frac{11}{16}$

5 Example 9 CAS An approximation for the area under the curve $y=\frac{1}{2}x^2$ from $x=0$ to $x=2$ with strips a $\frac{1}{4}$ unit wide, using Simpson's rule is

A $\frac{5}{64}$ B $\frac{21}{128}$ C $\frac{4}{3}$ D $\frac{13}{64}$ E $1\frac{1}{2}$

6 CAS An approximation for the area under the curve $y=\frac{1}{2}\log_e(x)$ from $x=1$ to $x=4$ with strips a $\frac{1}{4}$ unit wide, using Simpson's rule is

A 1.265 B 1.582 C 1.273 D 1.908 E 1.272

7 CAS For the area under the curve $y=\log_{10}(3x)$ from $x=1$ to $x=6$:

a find an approximation using the midpoint rule with 10 strips
b find an approximation using the trapezoidal rule with 10 strips
c find an approximation using Simpson's rule with 10 strips
d find the area using the algebraic integral.
e Comment on your answers.

8 CAS For the area under the curve $y=2x^3-4x^2+9x+6$ from $x=2$ to $x=4$:

a find an approximation using the midpoint rule with 16 strips
b find an approximation using the trapezoidal rule with 16 strips
c find an approximation using Simpson's rule with 16 strips
d find the area using the algebraic integral.
e Comment on your answers.

Reasoning and communication

9 Derive the midpoint rule.

10 Three points are given by (a, h_1), $(a+w, h_2)$ and $(a+2w, h_3)$, where $h_1, h_2, h_3 > 0$.

Show that the area under the parabola passing through the three points is given by $\frac{w}{3}(h_1+4h_2+h_3)$.

11 Given that the area of a double strip is given by $\frac{w}{3}(a_{2i} + 4a_{2i+1} + a_{2i+2})$ [for $i = 0$ to $\frac{n}{2} - 1$], derive Simpson's rule.

9.04 THE EXPONENTIAL PROBABILITY DENSITY FUNCTION

You have previously worked with the probability density function, and should remember that the area under the curve equals 1.

IMPORTANT

The **probability density function**, or pdf, of a continuous random variable is the function that when integrated over an interval gives the probability that the continuous random variable, having that pdf, lies in that interval.

You can choose any function that satisfies the conditions to be a probability density function. The choice is really governed by how you want to use the distribution you obtain. In a pdf, you expect to find the area under the curve to be equal to 1.

IMPORTANT

The **exponential distribution** with parameter λ has the probability density function

$$f(t) = \begin{cases} \lambda e^{-\lambda t}, & \text{for } t \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

where $P(a \leq t \leq b) = \int_a^b f(t)dt$

You can apply the particular probability density function, $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$, using the exponential random variable with parameter $\lambda > 0$, and use the exponential random variables and associated probabilities to model data and solve problems.

In this section you will consider the **exponential distribution**, with probability density function $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$.

Since the exponential function $f(t) \geq 0$ for all values of t, to show that it is a probability density function you need only show that the integral is equal to 1.

For the integral $\int_{-\infty}^{\infty} f(t)dt$, where $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$,

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)dt &= \int_{-\infty}^{0} f(t)dt + \int_0^{\infty} f(t)dt \\ &= \int_{-\infty}^{0} 0dt + \int_0^{\infty} \lambda e^{-\lambda t}dt \\ &= 0 + \left[-e^{-\lambda t}\right]_0^{\infty} \\ &= 0 + \left[0 - (-1)\right] \\ &= 1\end{aligned}$$

QED

This is a distribution defined for all positive values of t. However, since it has a parameter λ, it is also a family of distributions, determined by the value of λ.

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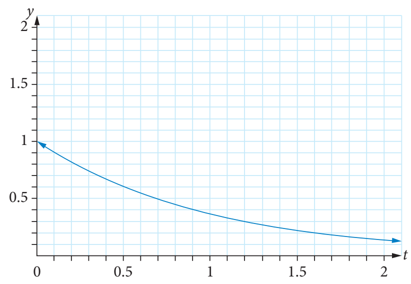
Example 10

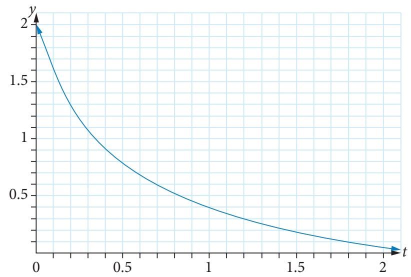
Draw graphs of the exponential distribution $f(t) = \lambda e^{-\lambda t}$ for $\lambda = 1$ and $\lambda = 2$, and comment on the shapes of the distributions.

Solution

Draw the graph of $f(t) = \lambda e^{-\lambda t}$ with $\lambda = 1$.



Draw the graph of $f(t) = \lambda e^{-\lambda t}$ with $\lambda = 2$.



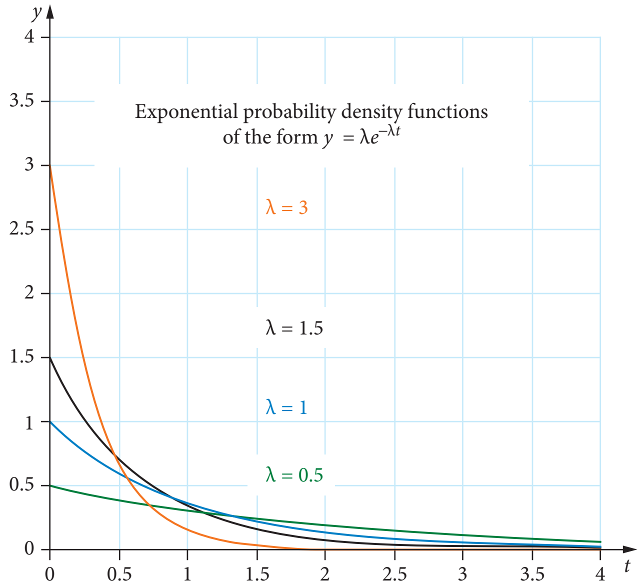
Compare the graphs.

For $\lambda = 2$, the distribution is twice as high at $t = 0$ as for $\lambda = 1$ and tapers to zero more quickly. This means that, for $\lambda = 2$, $f(t)$ has a higher probability of being close to zero.

Summarise the result.

The distribution with the higher value of λ starts higher and is skewed more to the left, so it has a higher probability of a lower value of t.

A summary of the effect of the λ value is below.



Remember that the rule for finding the probability of occurring in the interval $[a, b]$ is $P(a \le x \le b) = \int_a^b f(x)dx$

Example 11

A random variable X has an exponential distribution with $\lambda = 0.8$. Find $P(0 \le x \le 2)$.

Solution

Write the rule $P(a \le x \le b) = \int_a^b f(x)dx$.

$$P(0 \le x \le 2) = \int_0^2 f(x)dx$$

Write the rule for the exponential distribution with $\lambda = 0.8$.

$$f(x) = 0.8e^{-0.8x}$$

Substitute into the integral.

$$P(0 \le x \le 2) = \int_0^2 0.8e^{-0.8x}dx$$

Evaluate the integral.

$$= \left[-\frac{0.8}{0.8}e^{-0.8x}\right]_0^2$$

$$= -e^{-1.6} + e^0$$

$$= 1 - e^{-1.6}$$

Write the answer.

$$P(0 \le x \le 2) = 1 - e^{-1.6}$$

$$\approx 0.798$$

The exponential distribution is particularly important in calculating the probabilities of failures. You have seen in Maths Methods that the expected value of a continuous random variable X with probability density function $f(x)$ is given by $E(X) = \int_{-\infty}^{\infty} xf(x)dx$.

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IMPORTANT

Expected value of an exponential distribution

The expected value of an exponential distribution with parameter λ is given by $E(X) = \frac{1}{\lambda}$.

This is easily proven using integration by parts with $f(x) = x$ and $g(x) = -e^{-\lambda x}$ to give $\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$.

$$\begin{aligned}\int_0^\infty x\lambda e^{-\lambda x}\,dx &= \left[-xe^{-\lambda x}\right]_0^\infty - \int_0^\infty 1\cdot\left(-e^{-\lambda x}\right)dx \\ &= \left[-xe^{-\lambda x}\right]_0^\infty + \int_0^\infty 1\cdot\left(e^{-\lambda x}\right)dx \\ &= \left[-xe^{-\lambda x}\right]_0^\infty + \left[-\frac{1}{\lambda}e^{-\lambda x}\right]_0^\infty \\ &= (0-0) + \left[0-\left(-\frac{1}{\lambda}\right)\right] \\ &= \frac{1}{\lambda}\end{aligned}$$

This means that the exponential distribution with average value β has the parameter $\lambda = \frac{1}{\beta}$. For this reason, some people prefer to use the parameter $\frac{1}{\beta}$ in place of λ.

Example 12

The expected life (in hours) of a light bulb can be considered to be a random variable with an exponential distribution. For incandescent bulbs, $\lambda = \frac{1}{900}$.

a Find the probability that an incandescent bulb lasts at least 1000 h.

b Find the expected time that an incandescent bulb lasts for.

c Compact fluorescent bulbs (CFLs) generally last 8 times as long as incandescent bulbs. What percentage of fluorescent bulbs would be expected to last at least 1000 h?

d LED bulbs last 5 times as long as CFLs and are cheaper to run than either CFLs or incandescent bulbs. What percentage of LED bulbs will last longer than 1000 hours?

Shutterstock.com/Alix Kreil

Solution

a Write the required probability. $P(x \geq 1000)$ is needed.

Write the rule for a probability density function. $P(a \leq x \leq b) = \int_a^b f(x)\,dx$

Substitute values using $\lambda = \frac{1}{900}$, $f(x) = \frac{1}{900}e^{-\frac{1}{900}x}$ and $a = 1000$, $b = \infty$.

$$P(1000 \leq x) = \int_{1000}^{\infty} \frac{1}{900}e^{-\frac{1}{900}x}\,dx$$

Integrate.

$$= \left[-e^{-\frac{1}{900}x}\right]_{1000}^{\infty}$$

$$= 0 - \left(-e^{-\frac{1000}{900}}\right)$$

$$\approx 0.3292$$

Write the answer. The probability is about 0.329.

b Find the expected value. $E(X) = \frac{1}{\lambda} = 900$

Write the answer. The expected time that a light bulb will work for is 900 hours.

c Write $E(X)$ for fluorescent bulbs. Expected life $= 8 \times 900 = 7200$

Find λ for fluorescent bulbs. $\lambda = \frac{1}{7200}$

Find the probability that $1000 \leq X \leq \infty$.

$$P(1000 \leq X) = \int_{1000}^{\infty} \frac{e^{-x/7200}}{7200}\,dx$$

Find the integral.

$$= \left[-e^{-x/7200}\right]_{1000}^{\infty}$$

Evaluate.

$$= 0.8703\ldots$$

Write the answer. About 87% can be expected to last at least 1000 hours.

d Write $E(X)$ for LED bulbs. Expected life $= 5 \times 7200 = 36\,000$

Find λ for LED bulbs. $\lambda = \frac{1}{36\,000}$

Find the probability that $1000 \leq X \leq \infty$.

$$P(1000 \leq X \leq \infty) = \int_{1000}^{\infty} \frac{e^{-x/36\,000}}{36\,000}\,dx$$

Find the integral.

$$= \left[-e^{-x/36\,000}\right]_{1000}^{\infty}$$

Evaluate.

$$= 0.9726\ldots$$

Write the answer. About 97% can be expected to last at least 1000 hours.

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EXERCISE 9.04 The exponential probability density function

WS Exponential distributions

Concepts and techniques

1 Example 10 Draw graphs of the exponential probability density function $f(t) = \lambda e^{-\lambda t}$ for the following values of λ and compare the shapes of the functions.

a $\lambda = \frac{1}{3}$ b $\lambda = 1$ c $\lambda = 3$ d $\lambda = \frac{1}{30}$

2 Example 11 A random variable T has an exponential distribution with $\lambda = 1.6$. $P(0 \le x \le 2)$ equals

A $1 - e^{-\frac{16}{5}}$ B $e^{-\frac{16}{5}} - 1$ C $e^{\frac{16}{5}} - 1$ D 0.959 E $1 - e^{-\frac{8}{5}}$

3 A random variable T has an exponential distribution with $\lambda = 2$. $P(1 \le x \le 2)$ equals

A 0.117 B $\frac{e^4}{e^2 - 1}$ C $\frac{e^2 - 1}{e^4}$ D $e^4 - e^{-2}$ E $1 - e^{-4}$

4 A random variable T has an exponential distribution with $\lambda = 8$. $P(1 \le x \le 3)$ equals

A 0.0003 B $\frac{e^{16} - 1}{e^{24}}$ C $e^{-8} - e^{24}$ D $e^{-8} - e^{-16}$ E $1 - e^{-16}$

5 A random variable T has an exponential distribution with $\lambda = \frac{1}{100}$. $P(1 \le x \le 10)$ equals

A 0.085 B 0.0852 C $1 - e^{-0.1}$

D $e^{-\frac{1}{100}} + e^{-\frac{1}{10}}$ E $e^{-\frac{1}{100}} - e^{-\frac{1}{10}}$

6 Draw graphs of the exponential probability density functions for the following values, and comment on the shapes of the functions.

a $\lambda = 0.25$ and $\lambda = 2$ b $\lambda = 0.1$ and $\lambda = 0.2$ c $\lambda = 0.7$ and $\lambda = 0.9$

7 A random variable X has an exponential distribution with $\lambda = 0.6$. Find each of the following.

a $P(0 \le x \le 3)$ b $P(3 \le x \le 10)$ c $P(9.5 \le x \le 10.5)$ d $P(x \ge 10)$

8 A random variable T has an exponential distribution with $\lambda = 3$. Find each of the following.

a $P(0 \le t \le 0.3)$ b $P(0.4 \le t \le 0.6)$ c $P(t \ge 0.3)$ d $P(t \ge 0.5)$

9 Example 12 An exponential distribution has parameter $\lambda = 1.5$. What is the expected value of the distribution?

10 What is the parameter λ for an exponential distribution with an average value of 16?

Reasoning and communication

11 The expected life (in hours) of a printer drum can be considered to be a random variable with an exponential distribution, for which $\lambda = \frac{1}{200}$.

a Find the probability that a particular printer drum lasts at least 100 h.

b Find the expected time that a particular printer drum will last.

12 The average life of a standard car battery is known to be 1.5 years. Use an exponential distribution to model the life of standard car batteries to find, correct to 3 decimal places, the values of:

a λ

b the proportion of standard batteries that fail before 1.5 years

c the proportion of standard batteries that last for at least 4 years.

13 The average life of a 'calcium' car battery is known to be 2.5 years. Use an exponential distribution to model the life of 'calcium' car batteries to find (correct to 3 significant figures):

a the percentage of 'calcium' batteries that fail before 2.5 years

b the median life of a car battery

c the percentage of 'calcium' batteries that last for at least 4 years.

9.05 APPLICATIONS OF THE EXPONENTIAL PROBABILITY DENSITY FUNCTION

The exponential probability density function, $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$, with parameter $\lambda > 0$, has wide application in probabilities associated with lifetimes, failures and the occurrence of events at fixed average rates. It is worthwhile specifying the cumulative distribution function for an exponential probability density function.

IMPORTANT

The **cumulative distribution function** for an exponential probability function with parameter λ is given by $\text{cdf}(t) = 1 - e^{-\lambda t}$.

One of the most important properties of the exponential probability distribution is often called its 'lack of memory'. This is because the probability of occurrence within a fixed time interval is the same throughout the distribution, no matter how long it is from the start. It is the only continuous probability distribution with this property.

IMPORTANT

For events with an exponential probability density function, the probability of an event occurring in the interval $[p, p + s]$ is equal to the probability that it occurs in the interval $[0, s]$ for any values s and p.

In other words, the probabilities of occurrence in equal intervals are equal.

This can be proven using the cumulative density function parameter for the random variable with parameter λ.

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Clearly, the probability of an event occurring in the interval $[p, p + s]$ is conditional on it *not having occurred* in the interval $[0, p]$.

$$P(p \le x \le p + s \text{ given } x > p) = \frac{cdf(p+s) - cdf(p)}{1 - cdf(p)}$$
$$= \frac{(1 - e^{-\lambda(p+s)}) - (1 - e^{-\lambda p})}{1 - (1 - e^{-\lambda p})}$$
$$= \frac{e^{-\lambda p} - e^{-\lambda(p+s)}}{e^{-\lambda p}}$$
$$= \frac{e^{-\lambda p}(1 - e^{-\lambda s})}{e^{-\lambda p}}$$
$$= 1 - e^{-\lambda s}$$

But this is cdf(s). **QED**

Example 13

At a call centre, the occurrence of calls over time follows an exponential probability distribution with a parameter of $\lambda = 2$, where time is in minutes.

a What is the probability of getting a call in the next 3 minutes?

b What is the expected waiting time to get a call?

c What is the average rate of calls per minute?

Solution

a Write the cdf. — $\text{cdf}(t) = 1 - e^{-\lambda t}$

Substitute the values $t = 3$ and $\lambda = 2$. — $\text{cdf}(3) = 1 - e^{-2 \times 3}$

Evaluate. — $= 0.9975\ldots$

Write the answer. — The probability is about 99.75%

b Write the formula for the expected value. — $E(X) = \frac{1}{\lambda}$

Substitute $\lambda = 2$. — $= 0.5$

Write the answer. — The expected waiting time to the next call is half a minute.

c Use the inverse. — The expected waiting time is 0.5 minutes, so the average rate is 2 calls per minute.

Example **13** demonstrates the relationship between the average rate of occurrences, the parameter and the expected waiting time for an exponential probability distribution.

IMPORTANT

An exponential probability distribution of occurrences over time, with parameter λ, has an average rate of occurrence of λ and an expected waiting time of $\frac{1}{\lambda}$.

Conversely, if the rate of occurrence of an event is λ, the occurrence follows an exponential probability distribution with parameter λ.

This can be proven using the same method as shown in Example **13**.

Car batteries are generally guaranteed for 12 months or two years. Rechargeable batteries last for 500–800 discharge-recharge cycles. There are many other items that eventually fail, and the probability of failure over time or the number-of-use cycles is often an exponential distribution.

You are probably more interested in the probability that something will keep working than the probability that it will fail. This is called its **reliability** and is the complement of the probability of failure.

IMPORTANT

The **reliability function** $R(t) = 1 - P(\text{failure})$. The independent variable in a reliability function is usually time or the number of uses (cycles).

If the average rate of failure is λ, then $R(x) = e^{-\lambda x}$.

The reliability function $R(t)$ gives the probability that the item will keep working for time t (or for the number of times t).

Example 14

A resistor has a constant failure rate of 0.04 per hour.

a What is the probability that the resistor will work for 100 hours before it fails?

b If 100 resistors are tested, how many would be expected to fail in 25 hours?

Solution

a Use $R(x) = e^{-\lambda x}$ where $\lambda = 0.04$.	$R(x) = e^{-0.04x}$
Find the reliability at 100 hours.	$R(100) = e^{-0.04(100)} = e^{-4}$ $\therefore R(100) \approx 0.0183$
Write the answer.	Probability of working for 100 hours is 0.0183.
b Find the reliability.	$R(t) = e^{-\lambda t}$
Substitute in the values.	$= e^{-0.04 \times 25}$ ≈ 0.3679
Write the probability of failure, $1 - R(t)$.	Probability of failing ≈ 0.6321
Find the expected number of failures out of 100.	Expected number $= 100 \times 0.6321$ ≈ 63
Write the answer.	About 63 resistors will fail within 25 hours.

There are often cases where you want to know a **quantile** for an exponential distribution. Remember that a quantile t_α is the value of t such that the area under the curve below t is equal to α, so $P(-\infty \le t \le t_\alpha) = \alpha$. For an exponential probability function, $t \ge 0$, so the expression for a quantile becomes $P(0 \le t \le t_\alpha) = \alpha$.

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Example 15

The reliability of an aircraft altimeter is 0.998 for 10 h of operation. Assuming that the life of the instrument is exponential, for what period is the probability of failure less than 5%?

Solution

Write the reliability for 10 h.	$R(10) = 0.998$
Write the reliability function.	$R(t) = e^{-\lambda t}$
Substitute in the values.	$0.998 = e^{-\lambda \times 10}$
Solve for λ.	$\log_e(0.998) = -10\lambda$
Evaluate.	$\lambda = 0.0002...$
Find t such that $R(t) = 0.95$.	$e^{-0.0002...t} = 0.95$
Solve for t.	$-0.0002...t = \log_e(0.95)$
Divide by -0.0002.	$t = 256.2...$
Write the answer.	The probability of failure is less than 5% for about 256 h.

It is sometimes useful to know the variance and standard deviation for an exponential distribution. You should remember from Maths Methods that for a continuous probability density function defined on the interval $[a, b]$, $Var(X) = \int_a^b p(x)(x-\mu)^2\,dx$, where μ is the expected value of X.

For an exponential probability density function with parameter λ, this gives

$$Var(X) = \int_0^\infty \lambda e^{-\lambda x}\left(x - \frac{1}{\lambda}\right)^2 dx$$

Using integration by parts *twice* gives $Var(X) = \frac{1}{\lambda^2}$, so the standard deviation is $\sigma = \frac{1}{\lambda}$.

You can calculate the median as $t_{0.5}$ using $0.5 = \text{cdf}(x) = 1 - e^{-\lambda t}$.

$e^{-\lambda t} = 0.5$, so the median is $\frac{-\log_e(0.5)}{\lambda} = \frac{\log_e(2)}{\lambda}$.

IMPORTANT

An exponential probability function with parameter λ has the following properties:

Expected value (*mean*) $\mu = \frac{1}{\lambda}$

Variance $\sigma^2 = \frac{1}{\lambda^2}$

Standard deviation $\sigma = \frac{1}{\lambda}$

Median is $\frac{\log_e(2)}{\lambda}$

EXERCISE 9.05 Applications of the exponential probability density function

WS Reliability

Concepts and techniques

1 Example 13 A battery has a constant failure rate of 0.02 per hour.

a The battery's reliability at 200 hours is

A e^{-4} B $e^{-0.02}$ C $e^{0.02}$ D 0.18 E 0.018

b If 50 batteries are tested, the number expected to be in a failed state after 20 hours is

A $1 - e^{-0.4}$ B $1 - e^{-4}$ C $e^{0.02}$ D 16 E 17

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2 A machine part has a constant failure rate of 0.2 per hour.

a The machine part's reliability at 100 hours is

A e^{-20} B $1 - e^{-20}$ C $1 - e^{-0.2}$ D $-e^{-0.2}$ E e^{-100}

b If 60 machine parts are tested, the number expected to be in a failed state after 10 hours is

A $1 - e^{-2}$ B e^{-2} C 51 D 52 E $1 - e^{-20}$

3 The life of an item that has a mean time to fail of 100 hours is exponentially distributed. The probability of surviving through the interval 0 to 20 hours is

A 0.8751 B $e^{-\frac{1}{5}}$ C 0.818 D $e^{-\frac{2}{5}}$ E $e^{\frac{1}{5}}$

4 The life of a hair dryer that has a mean time to fail of 90 hours is exponentially distributed. The probability of surviving through the interval 0 to 15 hours is

A 0.8751 B $e^{\frac{1}{6}}$ C 0.846 D $e^{-\frac{1}{6}}$ E $e^{\frac{1}{5}}$

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5 If you receive phone calls at an average rate of 3 per hour, how long can you expect to wait for the next call?

6 The average number of clicks on a Geiger counter is about 25 per minute.
  a How long can you expect to wait for the next click?
  b What is the standard deviation of the exponential distribution?
  c What is the median of the exponential distribution?
  d What is the first quartile?
  e What is the third quartile?

Reasoning and communication

7 Examples 14, 15 The average response time of students to teacher questions is about 14 seconds. Find the probability that the next question will be answered in:
  a i less than 8 s   ii more than 8 s
  b Comment on the total of these probabilities.

8 For an exponential distribution with parameter λ, find the value of λ such that $P(0 \le x \le \lambda) = 0.8$.

9 The time between successive phone calls at a call centre is known to follow an exponential distribution. 90% of the intervals between calls are less than 2 minutes. What percentage of the time intervals will be less than 5 minutes?

10 A computer-controlled machine lathe takes 4 minutes to complete a particular job. It completes this job with a reliability of 0.99. What will be its reliability if the time allowed is:
  a 3 min?   b 5 min?   c 10 min?

11 20% of batteries last longer than 4 years. Use an exponential model to find the guarantee period that a manufacturer should set to ensure that no more than 20% of batteries fail within the guarantee period.

12 Light switches have an average life of 3000 cycles and failures follow an exponential pattern.
  a What is the parameter λ for failure?
  b What is the probability that a light switch can be switched on and off 5000 times before it fails?

13 The average time taken to find a spare part for a car at a spare parts distributor is about 6 minutes (if it is in stock). Use the exponential distribution to determine:
  a the proportion of spare parts that take less than 2 minutes to find
  b the median time
  c the probability that the next spare part will take longer than 10 minutes to find.

9

CHAPTER SUMMARY

APPLICATIONS OF INTEGRATION

- An area between two curves can be calculated in two ways:
  - as the difference between the areas under the two functions; or
  - as the area under the difference function.
- **Volume of a shape** $V = \int A(h)\,dh = \int A\,dh$
- **Volumes of solids of revolution**

  The volume of a solid of revolution generated by a curve in the *xy* plane is given by

  $V = \pi \int y^2 dx$ when rotated about the *x*-axis

  or $V = \pi \int x^2 dy$ when rotated about the *y*-axis.
- The **midpoint rule** is that

  $\int_a^b f(x)dx \approx w\sum_{i=1}^{n} f(a_i)$, where the interval $[a, b]$ is divided into n equal width strips of width w and the values $a_1, a_2, \ldots, a_n$ are the midpoints of the strips.
- The **trapezoidal rule** is that

  $$\int_a^b f(x)dx \approx \frac{w}{2}\left(f(a_0) + 2\sum_{i=1}^{n-1} f(a_i) + f(a_n)\right),$$

  where the interval $[a, b]$ is divided into n equal width strips of width w and values $a_0, a_1, \ldots, a_n$. It can also be written as

  $\int_a^b f(x)dx \approx \frac{w}{2}(E + 2M)$, where E is the sum of the values of $f(x)$ at the ends of the interval and M is the sum of the values at the intervening points $a_1, a_2, \ldots, a_{n-1}$.
- **Simpson's rule** is that

  $$\int_a^b f(x)dx \approx \frac{w}{3}\left(f(a_0) + 2\sum_{i=1}^{\frac{n}{2}-1} f(a_{2i}) + 4\sum_{i=1}^{\frac{n}{2}} f(a_{2i-1}) + f(a_n)\right),$$

  where the interval $[a, b]$ is divided into an even number n of equal width strips of width w and end values $a_0, a_1, \ldots, a_n$. It can be written as $\int_a^b f(x)dx \approx \frac{w}{3}(X + 2E + 4O)$,

  where X is the sum of the values of $f(x)$ at the ends of the interval, E is the sum of the values at the even-numbered intervening points $a_2, a_4, \ldots, a_{n-2}$ and O is the sum of the values at the odd-numbered intervening points $a_1, a_3, \ldots, a_{n-1}$.
- The **probability density function, pdf,** of a continuous random variable is the function that when integrated over an interval gives the probability that the continuous random variable having that pdf, lies in that interval.
- The exponential distribution with parameter λ has the probability density function

  $$f(t) = \begin{cases} \lambda e^{-\lambda t}, & \text{for } t \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

  where $P(a \leq t \leq b) = \int_a^b f(t)\,dt$
- **Expected value of a continuous probability distribution**

  For a continuous random variable X with probability density function $f(x)$, the expected value of the random variable is given by

  $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$, provided the integral exists.

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- The exponential reliability function is

  $R(x) = e^{-\frac{x}{\theta}}$ for $x>0$

  $\therefore R(x) = e^{-\lambda x}$
- A **quantile** t_α is the value of t such that the area under the curve below t is equal to α, so $P(-\infty \le t \le t_\alpha) = \alpha$.
- The **expected value** of a continuous probability distribution for a continuous random variable X with probability density function $f(x)$, $E(X) = \int_{-\infty}^{\infty} xf(x)dx$, provided the integral exists.
- The exponential reliability function is $R(x) = e^{-\lambda x}$ for $x > 0$
- A **quantile** t_α is the value of t such that the area under the curve below t is equal to α, so $P(-\infty \le t \le t_\alpha) = \alpha$.
- An exponential random variable with parameter λ has the following properties:

  Expected value (*mean*) $\mu = \frac{1}{\lambda}$

  Variance $\sigma^2 = \frac{1}{\lambda^2}$

  Standard deviation $\sigma = \frac{1}{\lambda}$

  Median is $\frac{\log_e(2)}{\lambda}$

9

CHAPTER REVIEW
APPLICATIONS OF INTEGRATION

Multiple choice

1 Example 2 The area enclosed by $y = 2x^2 - x - 6$ and $y = x^2 + 2x + 4$ is:

A $93\frac{1}{3}$ B $57\frac{1}{6}$ C $36\frac{1}{6}$ D $5\frac{5}{24}$ E $7\frac{7}{24}$

2 Example 4 The formula for the volume of an equilateral triangular-based pyramid of side s and height h is

A $V = \frac{1}{2}bsh$ B $V = s^3h$ C $V = \frac{\sqrt{3}}{12}s^3h$

D $V = \frac{\sqrt{3}}{4}s^2h$ E $V = \frac{\sqrt{3}}{12}s^2h$

3 Example 7 An approximation for the definite integral of $f(x) = 2\sqrt{x-1}$ from $x = 1$ to $x = 5$ by calculating the areas of strips 1 unit wide, using the midpoint rule is closest to

A 10.77 square units B 5.16 square units C 15.01 square units

D 4 square units E 9.71 square units

4 Example 9 An approximation for the area under the curve $y = \frac{3}{4}x^2 + 1$ from $x = 1$ to $x = 5$ with strips 1 unit wide, using the trapezium rule is

A $\frac{35}{8}$ B $\frac{71}{2}$ C $\frac{139}{4}$ D $\frac{53}{2}$ E $\frac{89}{2}$

5 Example 11 A random variable T has an exponential distribution with $\lambda = 0.6$. $P(0 \le x \le 3)$ equals

A $1 - e^{-0.6}$ B $e^{-\frac{9}{5}} - 1$ C $e^{\frac{16}{5}} - 1$ D 0.835 E $1 - e^{-\frac{9}{5}}$

Short answer

6 Example 1 Find the area enclosed by $f(x) = x^2 - 2x$ and $g(x) = -x^2 + x$.

7 Example 2 Find the area enclosed by $f(x) = 3x^2 - 2x - 5$ and $g(x) = 2x^2 + 3x + 9$.

8 Example 3 Find the area enclosed by $f(x) = x^3 - 5x^2 - x + 13$ and $g(x) = 3x - 7$.

9 Example 5 Use a suitable definite integral to find the volumes, in cubic units, of solids formed by rotating the following curves about the x-axis between the limits shown.

a $y = x^2$, from $x = 0$ to $x = 3$ b $y = 2e^{2x}$, from $x = 1$ to $x = 4$

10 Example 6 Use a suitable definite integral to find the volumes, in cubic units, of solids formed by rotating the following curves about the y-axis between the limits shown.

a $y = x^2$, from $y = 0$ to $y = 3$ b $y = \log_e(2x)$, from $y = 0$ to $y = 1$.

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11 Example 8 CAS Use the trapezoidal rule to find the area under the curve $f(x) = 2e^x$, from $x = 1$ to $x = 6$, using strips of width 0.5 units.

12 Example 10 Draw graphs of the exponential probability density function $f(t) = \lambda e^{-\lambda t}$ for the following values of λ and compare the shapes of the functions.

a $\lambda = \frac{1}{2}$

b $\lambda = 2$

13 Find the expected value of a uniform distribution on the interval:

a [0, 30]

b [10, 50].

Application

14 Example 14 What is the parameter λ for an exponential distribution with an average value of 16?

15 Example 13 A car radio has a constant failure rate of 0.01 per hour.

a What is the car radio's probability of lasting at least 200 hours?

b If 100 radios are tested, how many would be expected to fail within 250 hours?

16 Examples 14, 15 The average response time *in seconds* of students to exam questions is an exponential distribution with $\lambda \approx 0.08$. Find the probability that the next question will be answered in:

a i less than 9 s   ii more than 9 s

b Comment on the total of these probabilities.

17 The Lorenz curve of income distribution for a small country is approximately

$$y = 0.2x^4 + 0.4x^3 + 0.4x^2$$

a Sketch the Lorenz curve.

b What proportion of total income is earned by the poorest 20% of the population?

c What is the coefficient of income inequality?

18 Find the area enclosed by $y = x^4 - 8x^2 + x + 6$ and $y = 2x^2 + x - 3$.

19 For an exponential distribution with parameter λ, find the value of λ such that $P(0 \le x \le \lambda) = 0.9$.

20 A modern aircraft jet engine will operate for an average of 2000 flights before needing to be shut down in flight. This may not even be noticed by passengers because of safety margins and pilot training. What is the probability that an engine will last for 3000 flights without problems?

Practice quiz